Quantum effects on criticality of an Ising model in scale-free networks: Beyond mean-field universality class

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We study the effect of quantum fluctuations on the critical behavior of the Ising ferromagnetic phase transitions that do *not* belong to the mean-field universality class. A model system is considered, in which Ising spins are placed on the nodes of a scale-free network. Our Monte Carlo analysis shows that the critical exponents differ from those of mean-field phase transitions when degree exponent γ is in the range $3<\gamma<5$. This confirms earlier analytic calculations based on ansatzes and approximation methods. As we apply quantum fluctuations by means of a magnetic field perpendicular to the Ising spin direction, the transition temperature T_c decreases with increasing magnetic field strength. We find, however, that the quantum fluctuations do not alter the critical exponents and the universality class remains unchanged.

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I. INTRODUCTION

The Ising model is one of the most popular models used for the study of collective behavior and critical phenomena. It is simple enough to provide clear physical insights, yet its behavior is very rich and complex. As in all other systems, the critical behavior of an Ising system is independent of microscopic details of the model, and its universality class is determined by the topology and symmetry of the system.

Recently, there has been much interest in the study of Ising systems where the spins are placed on nodes of complex networks, such as scale-free [1–3] and small-world networks [4–6]. In those novel systems, ferromagnetically interacting spins are in an ordered phase at low temperatures and may undergo a ferromagnetic-paramagnetic second-order phase transition as temperature increases. In Watts-Strogatz-type small-world networks [7], this phase transition is characterized by the typical critical exponents of the mean-field universality class. Even more interesting is the Ising model in scale-free networks. One of the important characteristics of a network is the degree distribution function $P(k)$, where k is the number of links of a node. For a scale-free network, it takes a power-law form $P(k)\propto k^{-\gamma}$ for large k. Analytic calculations based on a replica method with the replica symmetric ansatz [1] and an approximation of local treelike network [2] have been previously performed to obtain the transition temperature and the critical exponents of the Ising model in scale-free networks. They have predicted that the transition temperature depends on the moments of the degree distribution. This has also been confirmed by a Monte Carlo simulation calculation [3]. On the other hand, the critical exponents are independent of details of the network and is solely determined by the degree exponent γ. For example, the transition is of a mean-field type, if $\gamma>5$. However, if $3<\gamma<5$, the critical exponents take nontrivial values that depend only on k. The exponents have been analytically calculated in Refs. [1,2] within their ansatz and approximation schemes. Furthermore, if $\gamma\leq 3$, there is no phase transition in the thermodynamic limit and the spin system is always ferromagnetically ordered at all finite temperatures [1–3]. The first part of this Brief Report is devoted to numerical calculations of the critical exponents when $3<\gamma<5$. It would provide a nontrivial test of the validity of the methods used in the analytic calculations [1,2].

In another line of research, there have been attempts to apply quantum fluctuations to Ising systems in complex networks [8,9]. One motivation for studying the effect of quantum fluctuations in those systems comes from the recent interest in quantum computing. A quantum computer consists of spins (or pseudospins derived from two-state systems) interacting with one another through network [10]. It is important to study the effect of quantum fluctuations because in random networks, it has been shown that imperfect control of local quantum spins may cause computation errors that grow fast with the number of qubits [11]. Since quantum fluctuations tend to weaken ferromagnetic ordering, the critical temperature is expected to get reduced in their presence.

Another motivation comes from the following nontrivial question: other than topology and symmetry, what affects the universality class of critical behavior? Early studies of Ising model in small-world and scale-free networks have shown that if the universality class is a mean-field type, the critical exponents are robust even in the presence of quantum fluctuations [8,9]. Yet whether quantum fluctuations would affect the critical exponents of nonmean-field universality classes is a nontrivial question and has yet to be tested. We will try to answer this question in the second part of this Brief Report.

II. NUMERICAL TEST OF CRITICAL EXPONENTS

What define our scale-free network are the degree exponent γ, the number of nodes N, and the average degree $k_{\rm av}$. The degree distribution is given by

$$P(k)=\begin{cases}0, & \text{if } k<k_{\min}\\ P_0, & \text{if } k=k_{\min}\\ ck^{-\gamma}, & \text{if } k>k_{\min}\end{cases}. \qquad (1)$$

The parameters $k_{\min}$, P_0, and c are determined from the following conditions:

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$$\sum_{k=1}^{\infty} P(k) = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} kP(k) = k_{\text{av}}. \tag{2}$$

Because $k_{\min}$ must be an integer, we introduced a continuous parameter P_0 to ensure that the above conditions are satisfied for any value of the continuous variable k_{av}. The value of P_0 is set to be no greater than but as close as possible to $ck_{\min}^{-\gamma}$. Once $P(k)$ is determined, we construct our network in the following way. For each node, we probabilistically determine its degree k according to the distribution $P(k)$. Although the total number of degrees is not fixed in this method, it approaches Nk_{av} as we perform the simulation over many networks. Now imagine that each node carries as many "arms" as its degree. Then we connect all nodes to a single cluster by repeatedly making links between two randomly chosen arms, one of which is already in the cluster and the other not yet in the cluster. After all nodes are connected to the cluster, any remaining unconnected arms are randomly connected with one another until none is left [12]. In the whole process, we have made sure that no two nodes are connected by more than one link and no link connects a node to itself. We have checked that the joint probability $P(k,k')$ for degrees of directly linked nodes satisfies the condition for uncorrelated networks; i.e., $P(k,k')=kk'P(k)P(k')/\langle k\rangle^2$ [13].

The Ising Hamiltonian in this scale-free network is defined as

$$H = -\sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z \tag{3}$$

where σ_i^z is the z component of the Pauli-spin matrix at node i. Two spins interact with each other if and only if they are connected in the network. In terms of coupling constant, it may be written as

$$J_{ij} = \begin{cases} J, & \text{if nodes } i \text{ and } j \text{ are connected} \\ 0, & \text{otherwise} \end{cases}. \tag{4}$$

For simplicity, we have used a constant ferromagnetic coupling ($J>0$) for all connected spin pairs.

For Monte Carlo simulations, we have used the Wolff cluster algorithm, which is known to be especially efficient near the critical temperature [14]. The ferromagnetic-paramagnetic phase-transition temperature T_c was determined via finite-size scaling. Following standard procedures [15], we computed the fourth-order Binder cumulant of magnetization m

$$U_N(T) = 1 - \frac{\langle m^4\rangle}{3\langle m^2\rangle^2} \tag{5}$$

for several systems with different numbers of spins N. Here, $\langle\cdots\rangle$ denotes both thermal and network averages. The number of Monte Carlo steps per network and the number of averaged network configurations were not predetermined, but the simulation was repeated until desired accuracies in all measured physical quantities were achieved. Typically, about $10^7\sim10^8$ cluster flips were performed per network for about 1000 networks for each parameter set in order to obtain the accuracy required in the current analysis. As shown in Fig. 1, all curves cross at one single point, which is a characteristic of a critical point. For $\gamma>3$, we find that there is always a transition between a low- temperature ordered (ferromagnetic) phase and a high-temperature disordered (paramagnetic) phase at finite T_c. All of our numerical results of T_c are consistent with earlier analytic formula $T_c=-2J/\ln(1-2\langle k\rangle/\langle k^2\rangle)$ [1,2]. For convenience, we will use the temperature scale in which $k_B=1$ throughout this Brief Report.

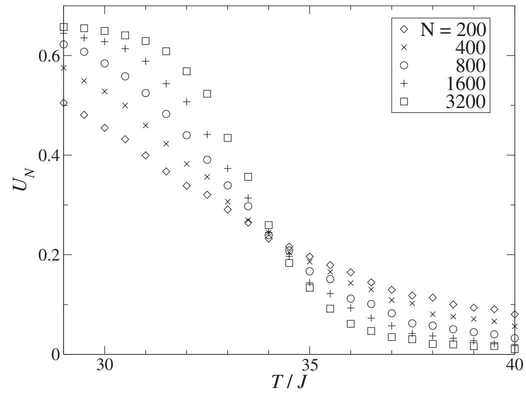

FIG. 1. The fourth-order Binder cumulant U_N as a function of temperature T is drawn for $\gamma=4.5$ and $k_{\text{av}}=30$. Error bars are about the same size as the symbols. The crossing point is interpreted as the critical point. In this example, $T_c/J=34.3\pm0.2$. This is consistent with the analytic formula [1,2] $T_c=-2J/\ln(1-2\langle k\rangle/\langle k^2\rangle)$, which becomes 34.35 ± 0.03 if we use our numerical results $\langle k\rangle=30.00\pm0.01$ and $\langle k^2\rangle=1060\pm1$.

The critical exponents are then obtained by fitting various physical quantities to the following finite-size scaling formulas:

$$U_N(T,N) = \tilde{U}((T-T_c)N^{1/\bar{\nu}}), \tag{6}$$

$$c(T,N) = N^{\alpha/\bar{\nu}}\tilde{c}((T-T_c)N^{1/\bar{\nu}}), \tag{7}$$

$$m(T,N) = N^{-\beta/\bar{\nu}}\tilde{m}((T-T_c)N^{1/\bar{\nu}}), \tag{8}$$

$$\chi(T,N) = N^{\bar{\gamma}/\bar{\nu}}\tilde{\chi}((T-T_c)N^{1/\bar{\nu}}), \tag{9}$$

where c, m, and χ are specific heat, magnetization, and magnetic susceptibility, respectively. Due to the infinite-range nature of the scale-free network, we introduced the exponent $\bar{\nu}$ which is related to the correlation volume (or correlation number) $N_c\propto|T-T_c|^{-\bar{\nu}}$ near the critical point. For a system with a finite dimension d, this would be given by $N_c=\xi^d$ where ξ is the correlation length. Since $\xi\propto|T-T_c|^{-\nu}$, we would get $\bar{\nu}=d\nu$. Note also that the exponent for the magnetic susceptibility has been denoted as $\bar{\gamma}$ in order to distinguish it with the degree exponent γ of the scale-free network.

The scaling functions are plotted in Fig. 2 for $\gamma=4.5$ and $k_{\text{av}}=30$. When scaled, data points from simulations of different system sizes all collapse to one single curve near T_c. The obtained critical exponents are compared with the results of analytic calculations in Table I. Our numerical results agree well with analytic results for all tested parameters [16]. Similar analyses have been performed for other values of γ as well. For example, we were able to confirm that our results

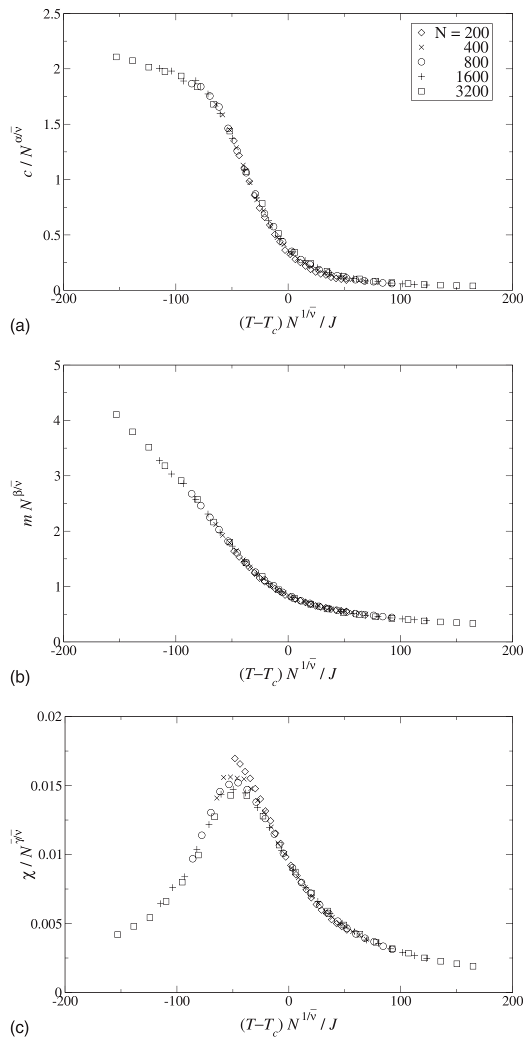

FIG. 2. Scaling functions for $\gamma=4.5$ and $k_{av}=30$. (a) Specific heat, (b) magnetization, and (c) magnetic susceptibility are plotted using $T_c=34.3\pm0.2$. Error bars are about the same as or smaller than the symbol sizes. The critical exponents are determined to be $\alpha=-0.3\pm0.1$, $\beta=0.67\pm0.05$, $\bar{\gamma}=1.00\pm0.05$, and $\bar{\nu}=2.4\pm0.1$.

are consistent with analytic results for $\gamma=4.0$ and 4.8 as well [17].

III. EFFECT OF QUANTUM FLUCTUATIONS

Now we will introduce quantum fluctuations to the above Ising model by applying a magnetic field Δ perpendicular to the Ising spin direction. Below we will choose it to be in the x-direction. The Hamiltonian thus becomes

$$H=-\sum_{i<j} J_{ij}\sigma_i^z\sigma_j^z-\Delta\sum_i \sigma_i^x. \quad (10)$$

Note that σ_i^x does not commute with σ_i^z. The second term can flip local spins and this will tend to weaken ferromagnetic ordering of Ising spins.

In order to analyze this model, we will employ the quantum Monte Carlo simulation method used in Ref. [8]. In short, our quantum-mechanical problem is mapped to a classical problem of statistical mechanics with additional time dimension. Taking the trace in the partition function $Z(\beta)=\mathrm{tr}\exp(-\beta H)$, the state of each spin is represented by $|S_i(\tau)\rangle$, where $\tau(\equiv it)$ is the imaginary time and $\beta\equiv1/T$. Then the whole imaginary time range $0\le\tau<\beta$ is divided into M slices of equal size and the completeness relation $\Pi_i(\Sigma_{S_i=\pm}|S_i(\tau_j)\rangle\langle S_i(\tau_j)|)$ is inserted at each time step $\tau_j=(j/M)\beta$ where $j=0,\cdots,M-1$. Using a simple Trotter product formula, we obtain an effective interaction between spins in the time direction. This formalism becomes exact in the limit $M\to\infty$. In actual simulations, M must be finite, but we make it large enough that the result becomes independent of its value. The typical value of M used in this study was between 30 and 150. In general, larger Δ and smaller T require more time slices and more simulation time.

TABLE I. Our critical exponents near the ferromagnetic-paramagnetic phase transitions for $\gamma=4.5$ and $k_{av}=30$ are compared with analytic results. $\tau\equiv(T-T_c)/T_c$ is the reduced temperature. Only negative values of τ are considered for the critical behavior in m. The analytic result of $\bar{\nu}$ has been obtained from the hyperscaling relation $2-\alpha=d\nu=\bar{\nu}$.

Critical behavior	Our results	Analytic results[a]
$c\propto\|\tau\|^{-\alpha}$	$\alpha=-0.3\pm0.1$	$(\gamma-5)/(\gamma-3)=-1/3$
$m\propto(-\tau)^{\beta}$	$\beta=0.67\pm0.05$	$1/(\gamma-3)=2/3$
$\chi\propto\|\tau\|^{-\bar{\gamma}}$	$\bar{\gamma}=1.00\pm0.05$	1
$N_c\propto\|\tau\|^{-\bar{\nu}}$	$\bar{\nu}=2.4\pm0.1$	$2-\alpha=7/3$

[a] References [1,2].

Figure 3 shows the transition temperature T_c as a function of perpendicular magnetic field Δ for $\gamma=4.5$ and $k_{av}=30$. As Δ increases, T_c continuously decreases and seems to end at a quantum critical point Δ_c. Right at the quantum critical point, T would be zero $(\beta\to\infty)$ so that infinitely many imaginary time slices would be needed to analyze it with the above method. It is beyond the scope of this Brief Report to prove the existence of a quantum critical point [18].

The scaling functions have been again calculated for finite values of Δ. They are plotted in Fig. 4 for $\Delta=20J$. It is evident that the finite-size scaling hypothesis still works very

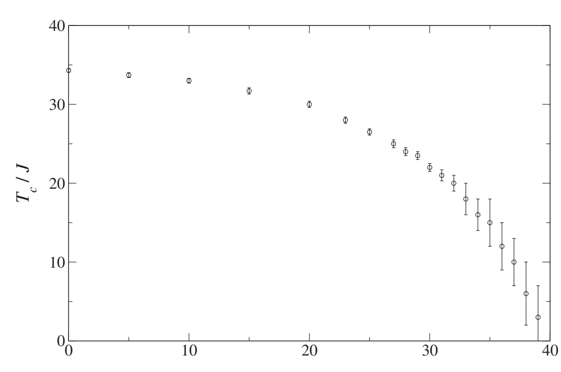

FIG. 3. Transition temperature T_c is drawn against the perpendicular magnetic field Δ. When interpolated, the curve separates the ferromagnetic and the paramagnetic phases. The parameters used in this plot are $\gamma=4.5$ and $k_{av}=30$.

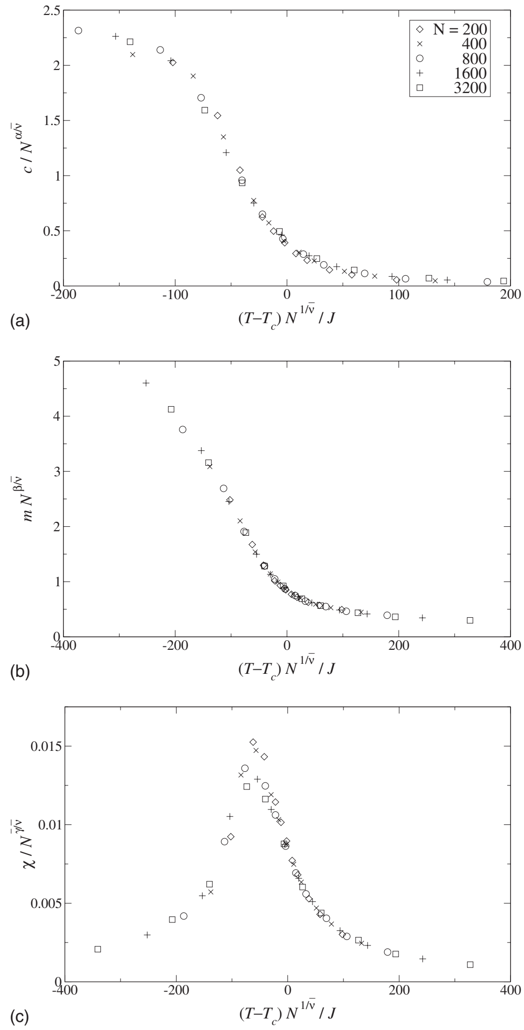

FIG. 4. Scaling functions for $\gamma=4.5$, $k_{\text{av}}=30$, and $\Delta=20J$. (a) Specific heat, (b) magnetization, and (c) magnetic susceptibility are plotted using $T_c=30.2\pm0.3$. Error bars are about the same size as the symbols. We find that the critical exponents are unchanged from the classical values in Table I: $\alpha=-0.3\pm0.1$, $\beta=0.66\pm0.07$, $\bar{\gamma}=1.00\pm0.05$, and $\bar{\nu}=2.3\pm0.1$.

well because all points collapse to the same curve near T_c. The critical exponents thus obtained are given by $\alpha=-0.3\pm0.1$, $\beta=0.66\pm0.07$, $\bar{\gamma}=1.00\pm0.05$, and $\bar{\nu}=2.3\pm0.1$. Within the error bars, these values are the same as those obtained without quantum fluctuations. We therefore conclude that even when the universality class is nonmean-field-like quantum fluctuations do not alter the critical exponents of the Ising model in scale-free network. We have also performed the same analysis for $\gamma=4.0$ and 4.8, obtaining similar results [17].

IV. SUMMARY AND DISCUSSION

We have numerically analyzed a scale-free network-connected Ising system when its critical behavior does not belong to the mean-field universality class. The critical exponents have been obtained from the Monte Carlo simulations. They agree well with analytic results, thus providing nontrivial verification of the ansatz and approximation used in the earlier analytic calculations.

By applying a magnetic field perpendicular to the Ising spin direction, we have also investigated the effect of quantum fluctuations to the critical behavior. Our numerical results show that quantum fluctuations reduce ferromagnetic-paramagnetic phase-transition temperature. However, they do not affect the critical exponents even for nonmean-field universality classes.

Although the analysis in this Brief Report is restricted to the range of degree exponent $3<\gamma<5$ where the classical model exhibits a phase transition, it would be interesting to see how quantum fluctuations affect the critical behavior for $\gamma\leq3$, where the classical model does not allow a disordered phase at any finite temperatures.

ACKNOWLEDGMENTS

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[16] Note that we have used $k_{\text{av}}=30$ for the data in Fig. 2. Since $k_{\text{av}}\gg2$, there are far more links than are needed for a treelike network. This implies that the approximation of local treelike structure, which has been used in one of the analytic calculations [2], may not be valid here. Nevertheless, both results are in good agreement.
[17] For the method used here, the smaller γ is, the longer simulation time it takes to get the same accuracy.
[18] In general, the time it takes for our method to achieve the same accuracy increases for smaller values of γ.